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【特约专稿】

时滞系统稳定性分析

——齐次多项式 Lyapunov 泛函方法

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摘 要:研究时变时滞线性系统的稳定性. 二次 Lyapunov 泛函是分析这类系统稳定性的有力工具,但往往难以给出低保守性稳定性条件. 为克服这个困难提出了齐次多项式 Lyapunov 泛函方法,建立了时滞系统的渐近稳定条件. 传统的二次 Lyapunov 泛函是所用泛函的特例. 数值例子表明所用方法有效,特别是时滞导数上界较大时,本方法效果显著,具有十分明显的低保守性优势.

关键词:时滞;齐次多项式 Lyapunov 泛函;Kronecker 积;稳定性

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Homogeneous polynomial Lyapunov functional for stability analysis of systems with delays

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Abstract: This paper addresses the stability problem of dynamic systems with time-varying delays. For this class of systems, quadratic Lyapunov functional is the most popular and powerful tool for stability analysis, and usually results in stability conditions with high conservativeness. In order to overcome the drawback, this paper tries to propose a new type of Lyapunov functional-homogeneous polynomial Lyapunov functional, based on which an asymptotic stability criterion is established. A numerical example shows that the presented method is effective, especially when the upper bounds of derivative of delays are relatively large.

Key words: delay; homogeneous polynomial Lyapunov functional; Kronecker product; stability

稳定性是动力学系统最重要的性质之一,在数学和工程领域得到广泛的研究[1-3]. 众所周知,Lyapunov 理论是分析稳定性最有效和最流行的工具,其核心是构造合适的 Lyapunov 函数(无时滞系统)[4-5]或 Lyapunov 泛函(时滞系统)[6-8].

由于计算技术不断发展,二次 Lyapunov 函数得到广泛应用,所得的稳定性判据一般用线性矩阵不等式来描述[9]. 对时滞系统,二次 Lyapunov 泛函是分析稳定性的有力工具[10-13]. 然而在很多情况下,用二次 Lyapunov 函数或二次 Lyapunov 泛函获得低保守性的稳定性判据相当不易[14-15]. 因此,需要寻求稳定性分析的新方法. 最

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近,协正多项式 Lyapunov 函数(齐次多项式 Lyapunov 函数的一种特殊形式)被用于任意切换信号的切换系统[16]. Chesi 等人提出一种无保守性线性矩阵不等式条件验证满足滞留时间的切换系统的指数稳定性[17],这启发了广大学者构造高次多项式 Lyapunov 函数,而不是二次 Lyapunov 函数,分析动力学系统的稳定性[17-20].

在此背景下,人们开始用多项式 Lyapunov 泛函研究时滞系统. 然而,多项式 Lyapunov 泛函方法尚未得到深入研究. 文献[21-22]尝试采用该方法建立时滞系统的稳定性条件. 需要注意的是,这两篇文献的主要推证有误. 因此,本文将进一步探索多项式 Lyapunov 泛函.

本文结构安排如下:第 1 节介绍了预备知识,第 2 节给出主要结果,第 3 节给出一个数值例子,第 4 节总结全文.

符号: $\mathbb{N}$ 为正整数集. 如 $m\in\mathbb{N}$, $\underline{m}=\{1,\cdots,m\}$. A^{T} 和 A^{-1} 分别为矩阵 A 的转置和逆. $\vec{A}=A+A^{\mathrm{T}}$. 符号"$*$"代表对称矩阵中的对称块. $A>0(<0)$ 表示 A 是正定(负定)矩阵. $\mathrm{diag}(a_1,\cdots,a_n)$ 是对角元为 $a_1,\cdots,a_n$ 的对角阵. $\mathbb{R}^{n\times m}$ 为 $n\times m$ 维的实矩阵, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$. 符号 $\mathbf{0}$ 为适当维数的零向量. $\mathbb{S}^n$ 是 $n\times n$ 维实对称矩阵集. $A\otimes B$ 表示 A 与 B 的 Kronecker 积, $A_{[q]}$ 为 A 的 q 次 Kronecker 积,即 $A_{[q]}=\underbrace{A\otimes A\otimes\ldots\otimes A}_{q}$. I_n 是 $n\times n$ 维单位矩阵. 给定 $[-d,a)$ 上的连续函数 $\boldsymbol{x}(s)$, $a>0,d>0$, $\forall t\in[0,a)$, $\boldsymbol{x}_t$ 是 $[t-d,t]$ 上的连续函数,定义为 $\boldsymbol{x}_t(\theta)=\boldsymbol{x}(t+\theta)$. $\|\boldsymbol{x}_t\|=\sup_{t-d\le s\le t}\{\|\boldsymbol{x}(s)\|\}$. $\mathcal{C}([a,b],X)$ 是从 $[a,b]$ 映射到 X 的连续函数. 如矩阵和向量的维数可从上下文推知,则不再指出.

1 问题陈述及预备知识

定义下面的符号方便使用:

$$\vartheta(n,q)=\frac{(n+q-1)!}{(n-1)!q!},\quad \omega(n,q)=\frac{1}{2}\vartheta(n,q)(\vartheta(n,q)+1)-\vartheta(n,2q).$$

给定正整数 q, $\boldsymbol{x}_{\{q\}}\in\mathbb{R}^{\vartheta(n,q)}$ 为一基向量,包含了 $\boldsymbol{x}$ 各分量所有 q 次单项式[23]. 给定 $\boldsymbol{x}_{\{q\}}$,存在列满秩矩阵 $K_q\in\mathbb{R}^{n^q\times\vartheta(n,q)}$ 使得 $K_q\boldsymbol{x}_{\{q\}}=\boldsymbol{x}_{[q]}$. 因 K_q 列满秩,存在其左逆矩阵 $\left(K_q^{\mathrm{T}}K_q\right)^{-1}K_q^{\mathrm{T}}$. 若 $q=0$ 则令 $\boldsymbol{x}_{[q]}=\boldsymbol{x}_{\{q\}}=1$. 线性空间 $L_{n,q}$ 定义如下:[23]

$$L_{n,q}=\left\{L\in\mathbb{S}^{\vartheta(n,q)}:\boldsymbol{x}_{\{q\}}^{\mathrm{T}}L\boldsymbol{x}_{\{q\}}=0\right\}.$$

下面的结果将反复使用.

引理 1 [24] 令 $A\in\mathbb{R}^{n\times m},B\in\mathbb{R}^{l\times k},C\in\mathbb{R}^{m\times q},D\in\mathbb{R}^{k\times p}$,则

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D).$$

引理 2 令 $\boldsymbol{x}_i\in\mathbb{R}^n,A_i\in\mathbb{R}^{n\times n},i\in\underline{m}$,有

$$\left(\sum_{i=1}^{m}A_i\boldsymbol{x}_i\right)_{[q]}=[A_1\ \ldots\ A_m]_{[q]}\left(\left[\boldsymbol{x}_1^{\mathrm{T}}\ \ldots\ \boldsymbol{x}_m^{\mathrm{T}}\right]^{\mathrm{T}}\right)_{[q]}.$$

计算直接表明如下事实:

事实 1 令 $\boldsymbol{x}\in\mathbb{R}^k,\boldsymbol{y}\in\mathbb{R}^n,\boldsymbol{z}\in\mathbb{R}^m$,有

$$(\boldsymbol{x}\otimes I_n\otimes\boldsymbol{z})\,\boldsymbol{y}=\boldsymbol{x}\otimes\boldsymbol{y}\otimes\boldsymbol{z}.$$

引理 3 [24] 令 $\boldsymbol{x}\in\mathbb{R}^k,A\in\mathbb{R}^{n\times m}$,则 $\boldsymbol{x}\otimes A=E_{k,n}(A\otimes\boldsymbol{x})$,其中 $E_{k,n}$ 为 Kronecker 交换矩阵.

引理 4 设 $\boldsymbol{z}\in\mathbb{R}^n,W\in\mathbb{R}^{n\times n},q\in\mathbb{N},K_{q-1}\in\mathbb{R}^{n^{q-1}\times\vartheta(n,q-1)}$ 满足 $K_{q-1}\boldsymbol{x}_{\{q-1\}}=\boldsymbol{x}_{[q-1]}$. 下式对 $\forall$

$j \in \{0, \ldots, q-1\}$成立:

$$\left(I_{n^j} \otimes W \otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]} \otimes \boldsymbol{z} \otimes \boldsymbol{x}_{[q-1-j]}\right) = \left(I_{n^j} \otimes E_{n,n^{q-1-j}}\right)\left(I_{n^{q-1}} \otimes W\right)\left(K_{q-1} \otimes I_n\right)\left(\boldsymbol{x}_{\{q-1\}} \otimes \boldsymbol{z}\right).$$

证明:由引理 3 可知,

$$\begin{aligned}
&\left(I_{n^j} \otimes W \otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]} \otimes \boldsymbol{z} \otimes \boldsymbol{x}_{[q-1-j]}\right) = \\
&\left(I_{n^j} \otimes W \otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]} \otimes \left(E_{n,n^{q-1-j}}\left(\boldsymbol{x}_{[q-1-j]} \otimes \boldsymbol{z}\right)\right)\right)= \\
&\left(I_{n^j} \otimes W \otimes I_{n^{q-1-j}}\right)\left(\left(I_{n^j}\boldsymbol{x}_{[j]}\right) \otimes \left(E_{n,n^{q-1-j}}\left(\boldsymbol{x}_{[q-1-j]} \otimes \boldsymbol{z}\right)\right)\right)= \\
&\left(I_{n^j} \otimes W \otimes I_{n^{q-1-j}}\right)\left(I_{n^j} \otimes E_{n,n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]} \otimes \left(\boldsymbol{x}_{[q-1-j]} \otimes \boldsymbol{z}\right)\right)= \\
&\left(I_{n^j} \otimes \left(\left(W \otimes I_{n^{q-1-j}}\right)E_{n,n^{q-1-j}}\right)\right)\left(\boldsymbol{x}_{[q-1]} \otimes \boldsymbol{z}\right)= \\
&\left(I_{n^j} \otimes \left(\left(W \otimes I_{n^{q-1-j}}\right)E_{n,n^{q-1-j}}\right)\right)\left(K_{q-1} \otimes I_n\right)\left(\boldsymbol{x}_{\{q-1\}} \otimes \boldsymbol{z}\right).
\end{aligned}$$

根据$E_{n,m}E_{m,n} = I_{nm}$[24],

$$\begin{aligned}
&I_{n^j} \otimes \left(\left(W \otimes I_{n^{q-1-j}}\right)E_{n,n^{q-1-j}}\right)= \\
&I_{n^j} \otimes \left(E_{n,n^{q-1-j}}\left(I_{n^{q-1-j}} \otimes W\right)E_{n^{q-1-j},n}E_{n,n^{q-1-j}}\right)= \\
&I_{n^j} \otimes \left(E_{n,n^{q-1-j}}\left(I_{n^{q-1-j}} \otimes W\right)\right)= \\
&\left(I_{n^j} \otimes E_{n,n^{q-1-j}}\right)\left(I_{n^{q-1}} \otimes W\right).
\end{aligned}$$

由上面两个方程可知引理成立.

2　主要结果

考虑系统

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{x}\left(t - \tau(t)\right), \quad t \geq 0, \\ \boldsymbol{x}(t) = \boldsymbol{\phi}(t), \quad t \in [-\tau, 0]. \end{cases} \tag{1}$$

其中$\boldsymbol{x}(t) \in \mathbb{R}^n$是状态变量,$A, B \in \mathbb{R}^{n\times n}$是系统矩阵,时滞$\tau(t)$满足$0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq h$,$\tau, h$是正常数,$\boldsymbol{\phi} \in \mathcal{C}\left([-d, 0], \mathbb{R}^n\right)$是向量值初始函数.

由事实 1 和文献[23],可知

$$\begin{aligned}
&K_q\frac{\mathrm{d}\boldsymbol{x}_{\{q\}}(t)}{\mathrm{d}t} = \\
&\frac{\mathrm{d}K_q\boldsymbol{x}_{\{q\}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}_{[q]}(t)}{\mathrm{d}t} = \frac{\partial\boldsymbol{x}_{[q]}(t)}{\partial\boldsymbol{x}(t)}\dot{\boldsymbol{x}}(t)= \\
&\left(\sum_{j=0}^{q-1}\boldsymbol{x}_{[j]}(t) \otimes I_n \otimes \boldsymbol{x}_{[q-1-j]}(t)\right)A\boldsymbol{x}(t) + \\
&\quad\left(\sum_{j=0}^{q-1}\boldsymbol{x}_{[j]}(t) \otimes I_n \otimes \boldsymbol{x}_{[q-1-j]}(t)\right)B\boldsymbol{x}\left(t - \tau(t)\right)= \\
&\sum_{j=0}^{q-1}\boldsymbol{x}_{[j]}(t) \otimes \left(A\boldsymbol{x}(t)\right) \otimes \boldsymbol{x}_{[q-1-j]}(t) + \sum_{j=0}^{q-1}\boldsymbol{x}_{[j]}(t) \otimes \left(B\boldsymbol{x}\left(t - \tau(t)\right)\right) \otimes \boldsymbol{x}_{[q-1-j]}(t).
\end{aligned}$$

其中K_q满足$K_q\boldsymbol{x}_{\{q\}}(t) = \boldsymbol{x}_{[q]}(t)$. 根据引理 4,

$$\begin{aligned}
&\frac{\mathrm{d}\boldsymbol{x}_{[q]}(t)}{\mathrm{d}t}= \\
&\sum_{j=0}^{q-1}\left(I_{n^j}\otimes A\otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]}(t)\otimes\boldsymbol{x}(t)\otimes\boldsymbol{x}_{[q-1-j]}(t)\right)+ \\
&\quad\sum_{j=0}^{q-1}\left(I_{n^j}\otimes B\otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]}(t)\otimes\boldsymbol{x}\left(t-\tau(t)\right)\otimes\boldsymbol{x}_{[q-1-j]}(t)\right)= \\
&\sum_{j=0}^{q-1}\left(I_{n^j}\otimes A\otimes I_{n^{q-1-j}}\right)\boldsymbol{x}_{[q]}(t)+ \\
&\sum_{j=0}^{q-1}\left(I_{n^j}\otimes B\otimes I_{n^{q-1-j}}\right)\left(\boldsymbol{x}_{[j]}(t)\otimes\boldsymbol{x}\left(t-\tau(t)\right)\otimes\boldsymbol{x}_{[q-1-j]}(t)\right).
\end{aligned}\tag{2}$$

定义

$$\begin{aligned}
\mathcal{A}_q&=K_q^{-1}\sum_{j=0}^{q-1}\left(I_{n^j}\otimes E_{n,n^{q-1-j}}\right)\left(I_{n^{q-1}}\otimes A\right)K_q\ ; \\
\mathcal{B}_q&=K_q^{-1}\sum_{j=0}^{q-1}\left(I_{n^j}\otimes E_{n,n^{q-1-j}}\right)\left(I_{n^{q-1}}\otimes B\right)\left(K_{q-1}\otimes I_n\right).
\end{aligned}\tag{3}$$

其中，$K_q^{-1}=\left(K_q^{\mathrm{T}}K_q\right)^{-1}K_q^{\mathrm{T}}$是$K_q$的左逆矩阵，$K_{q-1}$满足$K_{q-1}\boldsymbol{x}_{\{q-1\}}(t)=\boldsymbol{x}_{[q-1]}(t)$.

式(2)和式(3)表明

$$\frac{\mathrm{d}\boldsymbol{x}_{\{q\}}(t)}{\mathrm{d}t}=\mathcal{A}_q\boldsymbol{x}_{\{q\}}(t)+\mathcal{B}_q\left(\boldsymbol{x}_{\{q-1\}}(t)\otimes\boldsymbol{x}\left(t-\tau(t)\right)\right).$$

给出主要结果前，先定义下列将在后面使用的符号.

$$\boldsymbol{\xi}(t)=\left[\begin{array}{ccc}\dot{\boldsymbol{x}}^{\mathrm{T}}(t) & \boldsymbol{x}^{\mathrm{T}}(t) & \boldsymbol{x}^{\mathrm{T}}\left(t-\tau(t)\right)\end{array}\right]_{\{q\}}^{\mathrm{T}}\in\mathbb{R}^{\vartheta(3n,d)}.$$

$\Lambda_1\in\mathbb{R}^{\vartheta(n,d)\times\vartheta(3n,d)}$.满足$\dot{\boldsymbol{x}}_{\{q\}}(t)=\Lambda_1\boldsymbol{\xi}(t)$.

$\Lambda_2\in\mathbb{R}^{\vartheta(n,d)\times\vartheta(3n,d)}$满足$\boldsymbol{x}_{\{q\}}(t)=\Lambda_2\boldsymbol{\xi}(t)$.

$\Lambda_3\in\mathbb{R}^{\vartheta(n,d)\times\vartheta(3n,d)}$满足$\boldsymbol{x}_{\{q\}}\left(t-\tau(t)\right)=\Lambda_3\boldsymbol{\xi}(t)$.

$\Lambda_4\in\mathbb{R}^{\vartheta(2n,d)\times\vartheta(3n,d)}$满足$\boldsymbol{\eta}(t)=\Lambda_4\boldsymbol{\xi}(t)$.

$\Lambda_5\in\mathbb{R}^{\vartheta(n,d-1)\times\vartheta(3n,d)}$满足$\boldsymbol{x}_{\{d-1\}}(t)\otimes\boldsymbol{x}(t-\tau(t))=\Lambda_5\boldsymbol{\xi}(t)$.

这些常矩阵Λ_i很容易得到.

下面的定理是本文的主要结果：

定理 1 给定$q\in\mathbb{N}$. 若存在正定矩阵$P,Q,X_{22}\in\mathbb{R}^{\vartheta(n,q)\times\vartheta(n,q)}$, $X_{11}\in\mathbb{R}^{\vartheta(2n,q)\times\vartheta(2n,q)}$, 任意矩阵$X_{21}\in\mathbb{R}^{\vartheta(n,q)\times\vartheta(2n,q)}$, $G\in\mathbb{R}^{\vartheta(3n,d)\times\vartheta(n,d)}$, $L_1,L_2,L_3,L_4\in L_{n,q}$, $L_5\in L_{2n,q}$, $L_6\in L_{3n,q}$使得如下条件成立：

$$P+L_1>0,\quad X_{22}+L_2>0,\qquad Q+L_3>0,\quad X+L_5,L_4>0.\tag{4}$$

$$\Omega+\overrightarrow{GK_q^{-1}\left[-I\ A\ B\right]_{[q]}K}+L_6<0.\tag{5}$$

其中, $X=\begin{bmatrix} X_{11} & * \\ X_{21} & X_{22} \end{bmatrix}$;

$$\Omega=\overrightarrow{\Lambda_2^{\mathrm{T}}P\left(\mathcal{A}_q\Lambda_2+\mathcal{B}_q\Lambda_5\right)}+\tau\Lambda_1^{\mathrm{T}}X_{22}\Lambda_1+\tau\Lambda_4^{\mathrm{T}}X_{11}\Lambda_4+$$
$$\overrightarrow{\Lambda_2^{\mathrm{T}}X_{21}\Lambda_4}-\overrightarrow{\Lambda_3^{\mathrm{T}}X_{21}\Lambda_4}+\Lambda_2^{\mathrm{T}}Q\Lambda_2+(h-1)\Lambda_3^{\mathrm{T}}Q\Lambda_3.$$

则系统(1)渐近稳定.

证明:选择 Lyapunov-Krasovskii 泛函如下:

$$V=V_1+V_2+V_3+V_4. \tag{6}$$

其中,

$$V_1(t)=\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(t)P\boldsymbol{x}_{\{q\}}(t).$$

$$V_2(t)=\int_0^{\tau}(\tau-\alpha)\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(t-\alpha)X_{22}\dot{\boldsymbol{x}}_{\{q\}}(t-\alpha)\mathrm{d}\alpha.$$

$$V_3(t)=\int_0^{t}\int_{\alpha-\tau(\alpha)}^{\alpha}\boldsymbol{\eta}^{\mathrm{T}}(\alpha,s)X\boldsymbol{\eta}(\alpha,s)\mathrm{d}s\mathrm{d}\alpha.$$

$$V_4(t)=\int_{t-\tau(t)}^{t}\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(s)Q\boldsymbol{x}_{\{q\}}(s)\mathrm{d}s.$$

$$\boldsymbol{\eta}(\alpha)=\begin{bmatrix}\boldsymbol{x}(\alpha)\\ \boldsymbol{x}(\alpha-\tau(\alpha))\end{bmatrix}_{\{q\}},\quad \boldsymbol{\eta}(\alpha,s)=\begin{bmatrix}\boldsymbol{\eta}(\alpha)\\ \dot{\boldsymbol{x}}_{\{q\}}(s)\end{bmatrix}.$$

由式(4)中条件$P+L_1>0$,如$\boldsymbol{x}(t)\neq\mathbf{0}$, 则得

$$V_1(t)=\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(t)P\boldsymbol{x}_{\{q\}}(t)=\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(t)\left(P+L_1\right)\boldsymbol{x}_{\{q\}}(t)>0.$$

显然,$\boldsymbol{x}_{\{q\}}(t)=\mathbf{0}$当且仅当$\boldsymbol{x}(t)=\mathbf{0}$. 类似地,式(4)表明 Lyapunov-Krasovskii 泛函正定.

将$V(t)$沿(1)求导可得

$$\begin{aligned}\dot{V}_1(t)&=2\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(t)P\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_{\{q\}}(t)=\\ &2\boldsymbol{x}_{\{q\}}^{\mathrm{T}}(t)P\left(\mathcal{A}_q\boldsymbol{x}_{\{q\}}(t)+\mathcal{B}_q\left(\boldsymbol{x}_{\{q-1\}}(t)\otimes\boldsymbol{x}(t-\tau(t))\right)\right)=\\ &2\boldsymbol{\xi}^{\mathrm{T}}(t)\Lambda_2^{\mathrm{T}}P\left(\mathcal{A}_q\Lambda_2+\mathcal{B}_q\Lambda_5\right)\boldsymbol{\xi}(t)=\\ &\boldsymbol{\xi}^{\mathrm{T}}(t)\overrightarrow{\Lambda_2^{\mathrm{T}}P\left(\mathcal{A}_q\Lambda_2+\mathcal{B}_q\Lambda_5\right)}\boldsymbol{\xi}(t).\end{aligned} \tag{7}$$

由于

$$V_2=\int_{t-\tau}^{t}(\tau-t+s)\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)X_{22}\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)\mathrm{d}s=$$
$$\int_{t-\tau}^{t}(\tau+s)\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)X_{22}\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)\mathrm{d}s-t\int_{t-\tau}^{t}\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)X_{22}\dot{\boldsymbol{x}}_{\{q\}}^{\mathrm{T}}(s)\mathrm{d}s.$$

有

$$\dot V_2(t)=\tau\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(t)X_{22}\dot{\boldsymbol x}_{\{q\}}(t)-\int_{t-\tau}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{22}\dot{\boldsymbol x}_{\{q\}}(s)\mathrm ds= \tau\boldsymbol\xi^{\mathrm T}(t)\Lambda_1^{\mathrm T}X_{22}\Lambda_1\boldsymbol\xi(t)-\int_{t-\tau}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{22}\dot{\boldsymbol x}_{\{q\}}(s)\mathrm ds. \tag{8}$$

V_3的导数为

$$\dot V_3= \tau(t)\boldsymbol\eta^{\mathrm T}(t)X_{11}\boldsymbol\eta(t)+2\int_{t-\tau(t)}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{21}\boldsymbol\eta(t)\mathrm ds+\int_{t-\tau(t)}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{22}\dot{\boldsymbol x}_{\{q\}}(s)\mathrm ds\le \tau\boldsymbol\eta^{\mathrm T}(t)X_{11}\boldsymbol\eta(t)+2\left(\boldsymbol x^{\mathrm T}_{\{q\}}(t)-\boldsymbol x^{\mathrm T}_{\{q\}}(t-\tau(t))\right)X_{21}\boldsymbol\eta(t)+\int_{t-\tau}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{22}\dot{\boldsymbol x}_{\{q\}}(s)\mathrm ds= \boldsymbol\xi^{\mathrm T}(t)\left(\tau\Lambda_4^{\mathrm T}X_{11}\Lambda_4+\overrightarrow{\Lambda_2^{\mathrm T}X_{21}\Lambda_4}-\overrightarrow{\Lambda_3^{\mathrm T}X_{21}\Lambda_4}\right)\boldsymbol\xi(t)+\int_{t-\tau}^{t}\dot{\boldsymbol x}^{\mathrm T}_{\{q\}}(s)X_{22}\dot{\boldsymbol x}_{\{q\}}(s)\mathrm ds. \tag{9}$$

易见V_4的导数为

$$\dot V_4\le\boldsymbol x^{\mathrm T}_{\{q\}}(t)Q\boldsymbol x_{\{q\}}(t)+(h-1)\boldsymbol x^{\mathrm T}_{\{q\}}(t-\tau(t))Q\boldsymbol x_{\{q\}}(t-\tau(t))= \boldsymbol\xi^{\mathrm T}(t)\left(\Lambda_2^{\mathrm T}Q\Lambda_2+(h-1)\Lambda_3^{\mathrm T}Q\Lambda_3\right)\boldsymbol\xi(t). \tag{10}$$

由式(6)到式(10)可知：

$$\dot V=\dot V_1(t)+\dot V_2(t)+\dot V_3(t)+\dot V_4(t)\le\boldsymbol\xi^{\mathrm T}(t)\Omega\boldsymbol\xi(t). \tag{11}$$

其中，

$$\Omega=\overrightarrow{\Lambda_2^{\mathrm T}P\left(\mathcal A_q\Lambda_2+\mathcal B_q\Lambda_5\right)}+\tau\Lambda_1^{\mathrm T}X_{22}\Lambda_1+\tau\Lambda_4^{\mathrm T}X_{11}\Lambda_4+ \overrightarrow{\Lambda_2^{\mathrm T}X_{21}\Lambda_4}-\overrightarrow{\Lambda_3^{\mathrm T}X_{21}\Lambda_4}+\Lambda_2^{\mathrm T}Q\Lambda_2+(h-1)\Lambda_3^{\mathrm T}Q\Lambda_3.$$

方程(1)和引理2表明：

$$\mathbf 0=\left(-\dot{\boldsymbol x}(t)+A\boldsymbol x(t)+B\boldsymbol x(t-\tau(t))\right)_{\{q\}}= K_q^{-1}\left(A\boldsymbol x(t)+B\boldsymbol x(t-\tau(t))-\dot{\boldsymbol x}(t)\right)_{[q]}= K_q^{-1}[-I\ A\ B]_{[q]}\begin{bmatrix}\dot{\boldsymbol x}(t)\\ \boldsymbol x(t)\\ \boldsymbol x(t-\tau(t))\end{bmatrix}_{[q]}= K_q^{-1}[-I\ A\ B]_{[q]}K\begin{bmatrix}\dot{\boldsymbol x}(t)\\ \boldsymbol x(t)\\ \boldsymbol x(t-\tau(t))\end{bmatrix}_{\{q\}}= K_q^{-1}[-I\ A\ B]_{[q]}K\boldsymbol\xi(t).$$

因此，对任意$G\in\mathbb R^{\vartheta(3n,d)\times\vartheta(n,d)}$，下面的等式成立

$$\boldsymbol\xi^{\mathrm T}(t)GK_q^{-1}[-I\ A\ B]_{[q]}K\boldsymbol\xi(t)=0. \tag{12}$$

由式(11)和式(12)可知

$$\dot V\le\boldsymbol\xi^{\mathrm T}(t)\left(\Omega+GK_q^{-1}[-I\ A\ B]_{[q]}K\right)\boldsymbol\xi(t)= \boldsymbol\xi^{\mathrm T}(t)\left(\Omega+GK_q^{-1}[-I\ A\ B]_{[q]}K+L_6\right)\boldsymbol\xi(t).$$

由上式和式(5)可知$\dot V<0$. 证明完成.

3 数值例子

本节给出一个数值例子验证所得的理论结果.

考虑下面的系统方程:

$$\begin{cases}\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\tau(t)), & t \geq 0, \\ \boldsymbol{x}(t) = \boldsymbol{\phi}(t), & t \in [-\tau, 0].\end{cases}$$

其中$\boldsymbol{x}(k) = [x_1(k), x_2(k)]^{\mathrm{T}} \in \mathbb{R}^2$,

$$A = \begin{bmatrix}-2 & 0 \\ 1 & -3\end{bmatrix}, \quad B = \begin{bmatrix}-1 & 0 \\ -0.8 & -1\end{bmatrix}.$$

仿真结果见表 1. 由表 1 可得下面结论:

1) $\forall h$,定理 1 比文献[25]具有更低的保守性;

2) 若 h 比较小($h = 0.1, 0.3, 0.5, 0.6$),文献[26-27]结果保守性比定理 1 低;若 h 较大($h = 0.8, 0.9$),文献[26-27]保守性比定理 1 高许多;

3) 文献[26-27]对 h 的变化较敏感,定理 1 对 h 不敏感,说明本文的方法对快速变化的时滞有显著效果.

表 1　时滞的上界:时变时滞($q=2$)

Table 1　Upper bound of delays: Time-varying delays($q=2$)

时滞上界 \ 文献 h	文献[25]	文献[26]	文献[27]	定理 1
0.1	0.964	$\geqslant 3\times10^{14}$	$\geqslant 3\times10^{14}$	$\geqslant 6\times10^{10}$
0.3	0.874	$\geqslant 3\times10^{14}$	$\geqslant 3\times10^{14}$	$\geqslant 6\times10^{10}$
0.5	0.749	$\geqslant 3\times10^{14}$	$\geqslant 2\times10^{14}$	$\geqslant 6\times10^{10}$
0.6	0.666	$\geqslant 2\times10^{14}$	$\geqslant 2\times10^{12}$	$\geqslant 6\times10^{10}$
0.8	0.428	3.162	3.162	$\geqslant 5\times10^{10}$
0.9	0.249	1.581	1.584	$\geqslant 3\times10^{10}$

4 结论

本文针对时滞系统提出一种齐次多项式 Lyapunov 泛函方法,建立了系统的稳定性条件. 数值例子表明本文给出的方法对快速变化的时滞显著效果.

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